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Eigenvalue-counting methods for non-proportionally damped systems

J.S. Jo [a], H.J. Jung [b], M.G. Ko [c], I.W. Lee [a,*]

[a] *Structural Dynamics and Vibration Laboratory, Department of Civil and Environmental Engineering, Korea Advanced Institute of Science and Technology, 373-1 Kuseong-dong, Yuseong-gu, Daejeon 305-701, South Korea*

[b] *Department of Civil and Environmental Engineering, Sejong University, 98 Gunja-dong, Gwangjin-gu, Seoul 143-747, South Korea*

[c] *Department of Civil and Environmental Engineering, Kongju National University, 182 Shinkwan-dong, Kongju 314-701, Chungnam, South Korea*

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Abstract

This paper presents numerical methods of counting the number of eigenvalues for non-proportionally damped system in some interested regions on the complex plane. Most of the eigenvalue analysis methods for proportionally damped systems use the well-known Sturm sequence property to check the missed eigenvalues when only a set of the lowest modes is used. However, in the case of the non-proportionally damped systems such as the soil–structure interaction system, the structural control system and composite structures, no counterpart of the Sturm sequence property for undamped systems has been established yet. In this study, a numerical method based on argument principle is explained with emphasis on the discretization of the contour and a new method based on Gleyse's theorem is proposed. To verify the applicability of the methods, two numerical examples are considered.

Keywords: Eigenvalue-counting method; Multiplicity; Non-proportional damping; Sturm sequence property; Schur–Cohn matrix

1. Introduction

To obtain the dynamic response of a civil structure, it is economic and efficient to superpose the results of a few lowest modes. Therefore, there have been proposed many eigensolution techniques which can find only a set of the lowest modes. The Lanczos and subspace methods are belong to this type of technique. In these techniques, however, some important modes can be missed in the calculation process, because the methods do not calculate the complete eigenvector set of a structure. Hence, a checking technique for missed eigenvalues is required to find the missed one. In the case of a proportionally damped or real eigenvalue system, the well-known Sturm sequence property has hitherto been applied to check the missed eigenvalues (Meirovitch, 1980; Hughes, 1987; Petyt, 1990; Bathe, 1996).

* Corresponding author. Tel.: +82-42-869-5658/3618; fax: +82-42-864-3658/869-3658.
E-mail address: iwlee@kaist.ac.kr (I.W. Lee).

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In the case of the non-proportionally damped systems such as the soil–structure interaction system, the structural control system and composite structures, no counterpart of the Sturm sequence property for undamped systems has been developed yet (Newland, 1989). Hence, when some important modes are missed for those systems, it may lead to poor results in dynamic analysis. A number of researches (Rajakumar, 1991; Kim and Lee, 1999) have been performed to solve the eigenproblem with the damping matrix, whereas there have been few studies on a technique to count the number of eigenvalues in this case in the literature. Tsai and Chen (1993) proposed the extended Sturm sequence property that can determine the root distribution of a polynomial on some specified lines of the complex plane. However, this extended property cannot be applied to the non-proportionally damped system because it is very difficult to find the specified line of the complex plane in this case and the Sturm sequence cannot be formed by factorizing the considered matrix in the field of the complex arithmetic computation. Recently, Jung et al. (2001) proposed a numerical technique of checking missed eigenvalues for eigenproblem with damping matrix using argument principle. A complex-valued determinant function is defined and evaluated along a closed contour on the complex plane. By calculating the number of rotations of the defined function, we can obtain the number of eigenvalues in the closed contour.

In this paper, the argument principle-based method by Jung et al. (2001) is more clearly explained with emphasis on the discretization of the contour. In addition, a more systematic method based on Gleyse's theorem (1999) is proposed. The proposed method takes advantage of Rombouts' method to determine the characteristic polynomial of an eigenvalue problem and some standard numerical algorithms (Press et al., 1988) to factorize the Schur–Cohn matrix into its decomposed form $\mathbf{LDL}^{\mathrm{T}}$. In the method, we can determine the number of complex eigenvalues in some interested regions on the complex plane by counting the positive elements of the factorized diagonal matrix $\mathbf{D}$. To verify the applicability of the methods, two numerical examples are considered.

2. Complex eigenvalue problem

In the analysis of dynamic response of structural system, the equation of motion of damped systems can be written as:

$$\mathbf{M}\ddot{u}(t) + \mathbf{C}\dot{u}(t) + \mathbf{K}u(t) = 0, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $(n \times n)$ mass, non-classical damping and stiffness matrices, respectively, and $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ are the $(n \times 1)$ acceleration, velocity and displacement vectors, respectively. To find the solution of the free vibration of the system, we consider the following quadratic eigenproblem:

$$\lambda^2\mathbf{M}\phi + \lambda\mathbf{C}\phi + \mathbf{K}\phi = 0, \tag{2}$$

in which λ and ϕ are the eigenvalue and eigenvector of the system. There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether they correspond to overdamped or undamped modes.

In general, the mass matrix $\mathbf{M}$ is non-singular, that is $\det(\mathbf{M}) \neq 0$, and we can reformulate the quadratic system of equation to a state-space form by doubling the order of the system (Meirovitch, 1990; Rajakumar, 1993; Kim and Lee, 1999) such as:

$$\mathbf{A}\psi = \lambda\psi, \tag{3}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}, \quad \psi = \begin{Bmatrix} \phi \\ \lambda\phi \end{Bmatrix}. \tag{4}$$

Eq. (3) is a standard eigenproblem, and the form of the matrix **A** in Eq. (4) is widely used in control engineering field (Meirovitch, 1990).

3. Review of the argument principle-based counting method (Jung et al., 2001)

3.1. Argument principle for a characteristic polynomial

Using the relationship between the eigenvalues of an eigenproblem and the zeros of the corresponding characteristic polynomial, the eigenvalues of the quadratic eigenproblem as Eq. (2) are equal to the zeros of the following characteristic polynomial:

$$p(\lambda) = \det(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}) = a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0, \tag{5}$$

where λ is a complex value and a_i $(i = 0, 1, \ldots, 2n)$ the real coefficients. The value of λ that satisfies $p(\lambda) = 0$ is called an eigenvalue of the system.

If the characteristic polynomial $p(\lambda)$ is analytic in and on a simple closed contour S, the following argument principle can be applied:

$$N = \frac{1}{2\pi \mathrm{i}} \oint_S \frac{p'(\lambda)}{p(\lambda)}\,\mathrm{d}\lambda = \frac{\Delta\theta}{2\pi}, \tag{6}$$

where N is the number of zeros of $p(\lambda)$ in the contour S and $\Delta\theta$ is the variation of the argument θ of $p(\lambda)$ around the contour S.

Because $\Delta\theta/2\pi$ in the right side of Eq. (6) can be interpreted as the number of rotations, the characteristic polynomial $p(\lambda)$ that maps a moving point λ describing the contour S into a moving point $p(\lambda)$ encircles the origin of the $p(\lambda)$-plane N times if the polynomial $p(\lambda)$ has N zeros in the contour S in the λ-plane. An example that the characteristic polynomial $p(\lambda)$ has two zeros in a contour S, is shown in Fig. 1.

However, since it is difficult to directly evaluate Eq. (6) using the symbolic algebraic operations, the numerical, or the iterative, approach was developed to apply the aforementioned argument principle to the non-proportionally damped systems.

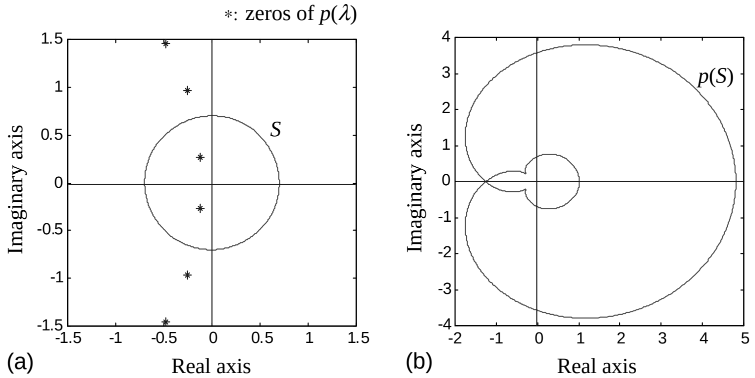

Fig. 1. Argument principle: (a) λ-plane; (b) $p(\lambda)$-plane.

3.2. Discretization of the contour S

The characteristic polynomial $p(\lambda)$ in Eq. (5) at point $\lambda = \lambda_j$ can be factorized as follows:

$$p(\lambda_j) = \bar{a}\prod_{i=1}^{2n}(\lambda_j - z_i) = r_j\angle\theta_j, \tag{7}$$

where $\bar{a}$ is a constant, z_i is the ith zero of the $p(\lambda)$, and r_j and θ_j the magnitude and argument of the value $p(\lambda_j)$ in polar form, respectively.

If we consider λ_j and z_i as a vector on the complex, $\lambda_j - z_i$ is also a vector from z_i to λ_j. The polar form of $\lambda_j - z_i$ can be written as:

$$\lambda_j - z_i = r_{j,i}e^{i\theta_{j,i}}, \tag{8}$$

where $r_{j,i}$ is the length and $\theta_{j,i}$ is the argument of the $\lambda_j - z_i$.

Using the polar form of $\lambda_j - z_i$ as in Eq. (8), the $p(\lambda_j)$ in Eq. (7) can be evaluated as:

$$p(\lambda_j) = \bar{a}\prod_{i=1}^{2n}(\lambda_j - z_i) = \bar{a}r_{j,1}r_{j,2}\cdots r_{j,2n}e^{i(\theta_{j,1}+\theta_{j,2}+\cdots+\theta_{j,2n})}. \tag{9}$$

So, r_j and θ_j of the $p(\lambda_j)$ in Eq. (7) has the following relationships

$$\begin{aligned} r_j &= \bar{a}_j r_{j,1}r_{j,2}\cdots r_{j,2n} = \bar{a}\prod_{i=1}^{2n} r_{j,i}, \\ \theta_j &= \theta_{j,1} + \theta_{j,2} + \cdots + \theta_{j,2n} = \sum_{i=1}^{2n}\theta_{j,i}. \end{aligned} \tag{10}$$

To consider the effect of the discretization of the contour S, we evaluate the value of $p(\lambda)$ at two consecutive discrete points $j = k$ and $j = k + 1$ as:

$$p(\lambda_k) = \bar{a}\prod_{i=1}^{2n}(\lambda_k - z_i) = r_k\angle\theta_k, \tag{11}$$

$$p(\lambda_{k+1}) = \bar{a}\prod_{i=1}^{2n}(\lambda_{k+1} - z_i) = r_{k+1}\angle\theta_{k+1}, \tag{12}$$

then, the change of argument $\Delta\theta_{k+1,k}$ from $p(\lambda_k)$ to $p(\lambda_{k+1})$ can be written as:

$$\Delta\theta_{k+1,k} = \theta_{k+1} - \theta_k = \sum_{i=1}^{2n}(\theta_{k+1,i} - \theta_{k,i}). \tag{13}$$

Since the argument change $\Delta\theta_{k+1,k}$ in Eq. (13) is simply sum of the effects of each zero of the characteristic polynomial $p(\lambda)$, we consider only the simplest case of $2n = 1$ as shown in Fig. 2. Fig. 2(a) represents the case that a zero is in the closed contour S, and Fig. 2(b) the case that a zero is outside the closed contour S. When a zero is in the closed contour S, the sum of the argument change along discrete points on the S is 2π, and when a zero is outside the closed contour S, the sum of the argument change is 0. So, if we evaluate $p(\lambda)$ at some finite discrete point along the closed contour S, the sum of the change of the argument of $p(\lambda)$ divided by 2π is exactly equal to the number of zeros in the contour S.

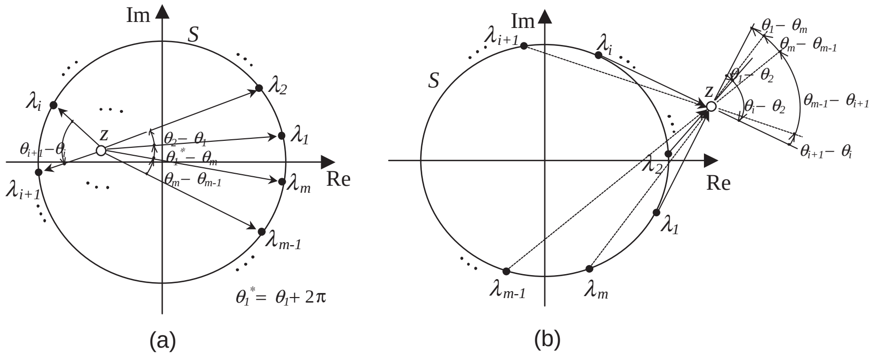

Fig. 2. Arguments at discrete points: (a) zero inside contour S; (b) zero outside contour S.

3.3. Evaluation of the arguments for $p(\lambda)$

The relationship between the characteristic polynomial and the factorized matrices by the $\mathbf{LDL}^{\mathrm{T}}$ factorization process can be used to evaluate the arguments for $p(\lambda)$. The contour S is considered as the set of the discrete checking and the $\mathbf{LDL}^{\mathrm{T}}$ factorization process is performed at each checking point. Then, the argument at each checking point can be calculated as follows (Korn and Korn, 1968; Pearson, 1974):

$$p(\lambda_j) = \det(\lambda_j^2\mathbf{M} + \lambda_j\mathbf{C} + \mathbf{K}) = \det\mathbf{LDL}^{\mathrm{T}} = \prod_{i=1}^{n} d_{ii} = r_j\angle\theta_j, \tag{14}$$

where d_{ii} is the diagonal elements of the diagonal matrix $\mathbf{D}$, and r_j and θ_j the magnitude and argument of the value $p(\lambda_j)$ in polar form, respectively. The number of the eigenvalues in the contour S is calculated by summing the variation of the argument of each checking point.

3.4. Considerations

In the implementation of the method for a practical problem, it is very important to properly choose the shape, the size and the number of discrete checking points of the closed contour S. The simplest shape of the contour is a disk of given radius about the origin. This shape can be applied to various damping cases such as underdamped, critically damped, and overdamped cases. Most of practical systems are underdamped and eigenvalues of the system are complex conjugate, so a half-circle and a line on the real axis are sufficient to check the missed eigenvalues in this case. Because the argument change along the real line is 0, the number of checking points for the half-circle and line on the real axis is about half of that for the complete circle.

The size of the contour, i.e., the radius of a half-circle should be only a little bit larger than the largest eigenvalue to be considered to ensure that the next largest eigenvalue is not within the contour. The size of the contour recommended is 1.005 times the magnitude of the largest eigenvalue. This size is also used in the proposed method in the following section.

The number of checking points recommended is six times the number of eigenvalues considered. After the contour is equally divided into checking points, the part of the contour close to the largest eigenvalue is subdivided because the argument jump occurs in the part of the contour close to an eigenvalue. And, if the drastic change of the variation of the argument between two adjacent checking occurs, the extra checking points between two adjacent checking points should be added.

The shape and size of the contour can be selected before the application of the counting processes. However, the number of checking points is varied during the processes when the drastic change of the variation of the argument occurs. And sometimes it is difficult to detect drastic change of the variation of the argument because the range of arguments is limited between 0° and 360° as well as it does not contain information about the number of rotations. If checking points are chosen sufficiently a lot, the missed eigenvalues can exactly be checked by the method. However, the $\mathbf{LDL}^{\mathrm{T}}$ factorizations of the characteristic polynomial at those points significantly increase the computational time.

4. Modified Sturm sequence property

4.1. Characteristic polynomial of a matrix

The characteristic polynomial in Eq. (5) can be obtained another way using the matrix $\mathbf{A}$ in Eq. (4) as:

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \hat{a}_{2n}\lambda^{2n} + \hat{a}_{2n-1}\lambda^{2n-1} + \cdots + \hat{a}_1\lambda + \hat{a}_0 = \sum_{i=0}^{2n} \hat{a}_i\lambda^i, \tag{15}$$

where λ is a complex value and $\hat{a}_i$ $(i = 0, 1, \ldots, 2n)$ are real coefficients. The coefficients a_i $(i = 0, 1, \ldots, 2n)$ in Eq. (5) are same scalar multiples to each $\hat{a}_i$ $(i = 0, 1, \ldots, 2n)$ in Eq. (15).

There are several methods for calculating the coefficients of the characteristic polynomial of a real square matrix. The most famous one is Faddeev–Leverrier's method (Faddeev and Faddeeva, 1953), which is often described as a standard method in text books (Chen, 1984; Franklin et al., 1998). Wang and Chen (1982) pointed out the numerical instability and inefficiency of Faddeev–Leverrier's method and proposed a numerically stable method to compute the characteristic polynomial based on Frobenius form of a matrix. This method needs to prescribe a small value to prevent some elements be divided by this small value and this value should be guided by error analysis and/or experience. Recently, Rombouts and Heyde (1998) presented an algorithm for calculating the coefficients of the characteristic polynomial of a general square matrix for the evaluation of canonical traces in determinant quantum Monte-Carlo methods. This algorithm does not include dividing operations, so it is stable and also known as efficient and accurate. In this paper, for calculating the coefficients of the characteristic polynomial of a matrix Rombouts algorithm is used.

A general real square matrix $\mathbf{A}$ of size $2n$-by-$2n$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,2n-1} & a_{1,2n} \\ a_{21} & a_{22} & \cdots & a_{2,2n-1} & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-1,1} & a_{2n-1,2} & \cdots & a_{2n-1,2n-1} & a_{2n-1,2n} \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2n-1} & a_{2n,2n} \end{bmatrix}, \tag{16}$$

can be transformed to upper Hessenberg form $\bar{\mathbf{A}}$:

$$\overline{\mathbf{A}} = \begin{bmatrix} \bar{a}_{1,1} & \bar{a}_{1,2} & \cdots & \bar{a}_{1,n-1} & \bar{a}_{1,n} \\ \bar{a}_{2,1} & \bar{a}_{2,2} & \cdots & \bar{a}_{2,n-1} & \bar{a}_{2,n} \\ 0 & \bar{a}_{3,2} & \cdots & \bar{a}_{3,n-1} & \bar{a}_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{a}_{n,n-1} & \bar{a}_{n,n} \end{bmatrix}, \tag{17}$$

Table 1
Rombouts' algorithm for calculating characteristic polynomial

Step 1: Reduce the given matrix $\mathbf{A}$ ($2n$-*by*-$2n$) to upper Hessenberg Form $\overline{\mathbf{A}}$.
- Use Householder reduction or Gauss-elimination like similarity transformations.

Step 2: Initialize a matrix $\mathbf{B}$ ($2n$-*by*-$2n$).
- Set all the elements of matrix $\mathbf{B}$ to 0.

Step3: Calculate elements b_{ij} *of the matrix* $\mathbf{B}$ *using the elements* $\bar{a}_{ij}$ *of the matrix* $\overline{\mathbf{A}}$ *as follows:*

```
DO j = 2n, 1, -1
    DO i = 1, j
        DO k = 2n - j, 1, -1
            b_{k+1,i} = ā_{i,j} b_{k,j+1} - ā_{j+1,} b_{k,i}
        ENDDO
        b_{1,i} = ā_{i,j}
    ENDDO
    DO k = 1, 2n - j
        b_{k,j} = b_{k,j} + b_{k,j+1}
    ENDDO
ENDDO
```

Step 4: Calculate the coefficients of the characteristic polynomial a_i $(i = 0, \dots, 2n)$
- Using the first row of the matrix $\mathbf{B}$, the coefficients of the characteristic polynomial can be computed as: $a_i = (-1)^{2n-i} b_{2n-i,1}$

by applying Householder reduction or sequence of Gaussian elimination like similarity transformations (Press et al., 1988). Because the matrix $\overline{\mathbf{A}}$ was obtained by applying similarity transformations to $\mathbf{A}$, the eigenvalues of both $\overline{\mathbf{A}}$ and $\mathbf{A}$ are same and the characteristic polynomials are scalar multiples to each other. For the purpose of calculating eigenvalues of the system, therefore, the characteristic polynomial can be considered as:

$$p(\lambda) = \det(\overline{\mathbf{A}} - \lambda\mathbf{I}). \tag{18}$$

If we define $\bar{p}(\lambda)$ as:

$$\bar{p}(\lambda) = \det(\mathbf{I} + \lambda\overline{\mathbf{A}}), \tag{19}$$

then this polynomial is closely related to $p(\lambda)$:

$$\bar{p}(\lambda) = (\lambda)^{2n} p(-1/\lambda). \tag{20}$$

The basic idea of the Rombouts' algorithm is to consider $\overline{\mathbf{A}} + \lambda\mathbf{I}$ as a matrix of polynomials in λ. We then calculate the polynomial $\bar{p}(\lambda)$ by evaluating the determinant in Eq. (19) using Gaussian elimination, with polynomials instead of scalars as matrix elements. As presented in Eq. (20) the coefficients of $\bar{p}(\lambda)$ are closely related to the coefficients of the $p(\lambda)$. The procedures of Rombouts' algorithm for calculating the coefficients of the characteristic polynomial of a $2n$-by-$2n$ matrix $\mathbf{A}$ are shown in Table 1.

4.2. Number of eigenvalues in an unit open circle

Gleyse and Moflih (1999) suggested a method of calculating the number of eigenvalues of a real polynomial in a unit open circle by a determinant representation.

Let $p(\lambda) = \sum_{h=0}^{2n} \hat{a}_h \lambda_h$ ($\hat{a}_h$ is a real number) be a characteristic polynomial of a given matrix $\mathbf{A}$, then the number of eigenvalues in a unit open circle can be determined as:

$$N_o = 2n - V[1, d_1, d_2, \dots, d_{2n}], \tag{21}$$

where N_o is the number of eigenvalues in a unit open circle, $2n$ is the degree of the polynomial, $V[k_0, k_1, k_2, \dots, k_{2n}]$ is the number of sign changes in the sequence k_i $(i = 0, 1, \dots, 2n)$ and d_i $(i = 1, 2, \dots, 2n)$ is the determinants (minors) of the leading principal submatrices of order i in the Schur–Cohn matrix $\mathbf{T}$:

$$\mathbf{T} = \begin{array}{c} \quad \mathbf{T}_1 \quad \cdots \quad \mathbf{T}_i \quad \cdots \quad \mathbf{T}_{2n} \\ \begin{bmatrix} t_{11} & \cdots & t_{1i} & \cdots & t_{1,2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ t_{i1} & \cdots & t_{ii} & \cdots & t_{i,2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ t_{2n,1} & \cdots & t_{2n,i} & \cdots & t_{2n,2n} \end{bmatrix} \end{array} \tag{22}$$

$$t_{ij} = \sum_{h=0}^{\min(i,j)} (\hat{a}_{2n-i+h}\hat{a}_{2n-j+h} - \hat{a}_{i-h}\hat{a}_{j-h}) \quad (i,j = 1,2,\ldots,2n),$$
$$d_i = \det(\mathbf{T}_i). \tag{23}$$

The processes of computing the number of eigenvalues in a unit open circle by the above method requires calculation of the characteristic polynomial of a given matrix **A**, the construction of the Schur–Cohn matrix **T** and the calculation of the determinants (minors) of the leading principal submatrices of order i in the Schur–Cohn matrix **T**. The coefficients of the characteristic polynomial of a given matrix can be determined by Rombouts' algorithm described at the previous chapter, and each element of the Schur–Cohn matrix can be obtained using Eq. (22).

4.3. Modified Sturm sequence property

Gleyse's theorem (1999) considers only about the number of eigenvalues in a unit open circle. To apply this theorem to an open circle of arbitrary radius ρ, we substitute $\lambda = \rho\bar{\lambda}$ (ρ is a real number) to Eq. (15), then the modified characteristic polynomial can be written as:

$$P(\bar{\lambda}) = \hat{a}_{2n}\rho^{2n}\bar{\lambda}^{2n} + \hat{a}_{2n-1}\rho^{2n-1}\bar{\lambda}^{2n-1} + \cdots + \hat{a}_1\rho\bar{\lambda} + \hat{a}_0 = \tilde{a}_{2n}\bar{\lambda}^{2n} + \tilde{a}_{2n-1}\bar{\lambda}^{2n-1} + \cdots + \tilde{a}_1\bar{\lambda} + \tilde{a}_0$$
$$= \sum_{i=0}^{2n} \tilde{a}_i\bar{\lambda}^i, \tag{24}$$

where $\tilde{a}_i = \hat{a}_i\rho^i$ $(i = 0,1,\ldots,2n)$ are modified coefficients.

Using the modified coefficients $\tilde{a}_i$ $(i = 0,1,\ldots,2n)$ in Eq. (24), this theorem can be extended to calculate the number of eigenvalues in an open disks of arbitrary radius.

The calculation of d_i $(i = 1,\ldots,2n)$ can be easily performed by the $\mathbf{LDL}^{\mathrm{T}}$ factorization of the Schur–Cohn matrix **T**. If $\mathbf{T} = \mathbf{LDL}^{\mathrm{T}}$, then:

$$\mathbf{T}_i = \mathbf{L}_i\mathbf{D}_i\mathbf{L}_i^{\mathrm{T}}, \tag{25}$$

$$\mathbf{L} = \begin{array}{c} \quad \mathbf{L}_1 \quad \cdots \quad \mathbf{L}_i \quad \cdots \quad \mathbf{L}_{2n} \\ \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ l_{i1} & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ l_{2n,1} & \cdots & l_{2n,i} & \cdots & 1 \end{bmatrix} \end{array}, \quad \mathbf{D} = \begin{array}{c} \quad \mathbf{D}_1 \quad \cdots \quad \mathbf{D}_i \quad \cdots \quad \mathbf{D}_{2n} \\ \begin{bmatrix} d_{11} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & d_{i,i} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & d_{2n,2n} \end{bmatrix} \end{array} \tag{26}$$

where the matrix $\mathbf{T}_i$ is the leading principal submatrices of order i in the Schur–Cohn the matrix **T**, the matrix $\mathbf{L}_i$ is the leading principal submatrices of order i in the factorized lower triangular matrix **L** and the

Table 2
Algorithm of the proposed method

Step 1: Change to a standard eigenproblem.
• Change the given eigenproblem to a standard form.
Step 2: Calculate the coefficients of the characteristic polynomial.
• Using Rombouts' algorithm, construct $p(\lambda) = \sum_{h=0}^{2n} a_h \lambda_h = 0$.
Step 3: Determine the radius ρ of an open disk.
• The radius $\rho(> 0)$ can be arbitrary, but select a little bit larger or smaller magnitude than the interested eigenvalue to minimize the possibility of existing unknown eigenvalues between them.
Step 4: Modify the coefficients of the characteristic polynomial.
• Substitute $\lambda = \rho\bar{\lambda}$ to $p(\lambda)$ obtained at step 2.
Step 5: Construct the Schur–Cohn matrix.
• Construct the Schur–Cohn matrix **T** using the modified coefficients of the characteristic polynomial.
Step 6: Perform the $\mathbf{LDL}^{\mathrm{T}}$ factorization the Schur–Cohn matrix **T**.
Step 7: Calculate the number positive elements in the matrix **D**.
• The number of eigenvalues inside the open circle is equal to the number positive elements in the matrix **D**.

matrix $\mathbf{D}_i$ is the leading principal submatrices of order i in the factorized diagonal matrix $\mathbf{D}$ as shown in Eq. (26). The value of d_i $(i = 1, \dots, 2n)$ can be evaluated as:

$$d_i = \det(\mathbf{T}_i) = \det(\mathbf{L}_i\mathbf{D}_i\mathbf{L}_i^{\mathrm{T}}) = \det(\mathbf{D}_i) = \prod_{h=1}^{i} d_{hh}. \tag{27}$$

Therefore, each $d_i = \det(\mathbf{T}_i)$ can be obtained by multiplying from the first diagonal element d_{11} to the ith diagonal element d_{ii} of the factorized diagonal matrix $\mathbf{D}$.

Considering Eq. (21), we only need to know the signs of each d_i because the unknown value of $V[1, d_1, d_2, \dots, d_{2n}]$ depends on sign changes of each d_i $(i = 1, \dots, 2n)$, and from Eq. (27) the sign change of d_i from d_{i-1} occurs when the diagonal element d_{ii} of the factorized diagonal matrix $\mathbf{D}$ is negative. So, the value of $V[1, d_1, d_2, \dots, d_{2n}]$ is equal to the number of negative element in the matrix $\mathbf{D}$. If we combine this result with Eq. (21), the number of eigenvalues in an open circle of radius ρ and the number of positive elements the factorized diagonal matrix $\mathbf{D}$ has the following relationship:

$$N_\rho = \text{the number of positive elements in } \mathbf{D}, \tag{28}$$

where N_ρ is the number of eigenvalues in an open circle of radius ρ and $\mathbf{D}$ is the diagonal matrix obtained by factorization of Schur–Cohn matrix $\mathbf{T}$ constructed using the modified coefficients in Eq. (24). This relation is very similar to the Sturm sequence property for undamped systems. The algorithm of the proposed method can be expressed as in Table 2.

5. Numerical examples

To show the applicability of the presented methods, two numerical examples are analyzed. A simple spring-mass-damper system that has the exact analytical eigenvalues is considered to verify that the methods can exactly calculate the number of eigenvalues in the open disk of arbitrary radius for the eigenproblem with the damping matrix. The plane frame structure with lumped dampers is also considered to verify the methods for the system with multiple eigenvalues.

5.1. Simple spring-mass-damper system (Chen, 1993)

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms:

$$\mathbf{M} = m\mathbf{I}, \tag{29}$$

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}, \tag{30}$$

$$\mathbf{K} = k\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \tag{31}$$

where α and β are the damping coefficients of the Rayleigh damping. The analytical solutions can be resulted through following relationships:

$$\lambda_{2i-1,2i} = -\xi_i\omega_i \pm \mathrm{j}\omega_i\sqrt{1-\xi_i^2} \quad \text{for } i = 1,\ldots,n, \tag{32}$$

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right), \tag{33}$$

$$\omega_i = 2\sqrt{\frac{m}{k}}\sin\frac{2i-1}{2n+1}\frac{\pi}{2}, \tag{34}$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively. A system with order 10 is used in analysis. k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0.05 and 0.5, respectively. All the eigenvalues and their radius from the origin in the complex plane are as in Table 3.

The number of considered eigenvalue is 16 and the radius of the contour S and the open circle is calculated by the 1.005 times the magnitude of the 16th eigenvalue ($\rho = 1.005|\lambda_{16}| = 1.8109$). The half-circle of the contour S is initially divided into 48 equal points as shown in Fig. 3. And since the argument of

Table 3
Calculated eigenvalues

Mode number	Eigenvalues (λ)		Radius ($\rho = \|\lambda\|$)
	Real	Imaginary	
1,2	−0.0306	±0.1463	0.1495
3,4	−0.0745	±0.4388	0.1495
5,6	−0.1585	±0.7133	0.4450
7,8	−0.2750	±0.9614	1.0000
9,10	−0.4137	±1.1763	1.2470
11,12	−0.5624	±1.3540	1.4661
13,14	−0.7077	±1.4932	1.6525
15,16	−0.8368	±1.5959	1.8019
17,18	−0.9381	±1.6651	1.9111
19,20	−1.0028	±1.7046	1.9777

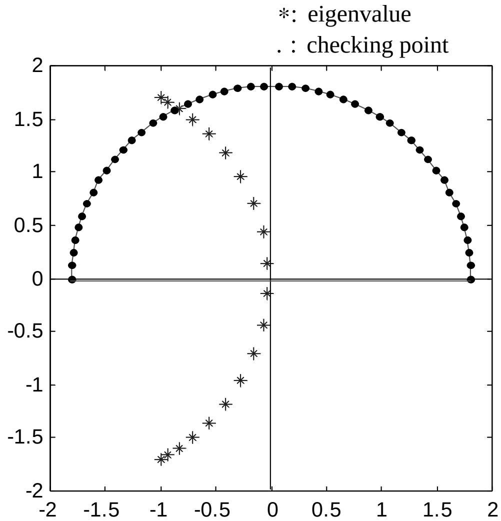

Fig. 3. Contour of S.

the largest eigenvalue is 117.67°, the part of the contour between 114.90° and 118.72° is subdivided into four equal parts. The total variation of the arguments is 2880°. And, the number of rotations is

$$N = \frac{\sum \Delta\theta_j}{2\pi} = \frac{2880^\circ}{360^\circ} = 8.$$

So, the number of eigenvalues in a circle of radius $\rho(= 1.005|\lambda_{16}|)$ is $16(= 8 \times 2)$, which exactly agree with the calculated values in Table 3. If we assume that the largest eigenvalue is unknown and use the same radius for the contour S, we can detect the checking point where the drastic change of the variation of the argument occurs in this case. The change of the variation of the arguments is 203.04° at checking point $\rho\angle 118.90^\circ$ Because this value is over 180°, we can conclude that new checking points are needed. The contour of $p(S)$ with 1000 checking points is shown in Fig. 4. It is very difficult to count the number of rotations using Fig. 4(a). Because only the arguments are important to count the number of eigenvalues, the magnitude of $p(S)$ can be scaled to help the graphical interpretation as shown in Fig. 4(b). The number of rotations for the contour $p(S)$ in the figure is 8.

The results for the modified Sturm sequence property are shown in Table 4. As shown at the last column in Table 5, the number of sign changes is 4. So if we use Eq. (21), the number of eigenvalues in the circle is $20 - 4 = 16$. Using Eq. (28), the number of positive elements in the matrix **D** is 16 as shown at the third column in Table 4. Therefore, we verify that the proposed method can exactly check the number of eigenvalues in some open disk of arbitrary radius.

5.2. Plane frame structure with lumped dampers (Kim et al., 1999)

In this example, a plane frame structure with lumped dampers is presented. The geometric configuration and material properties are shown in Fig. 5. The model is discretized in six beam elements with equal length for each direction resulting in the system of dynamic equation with a total of 18 degrees of freedom. Thus, the order of the associated eigenproblem is 36. The consistent damping matrix is derived from the classical

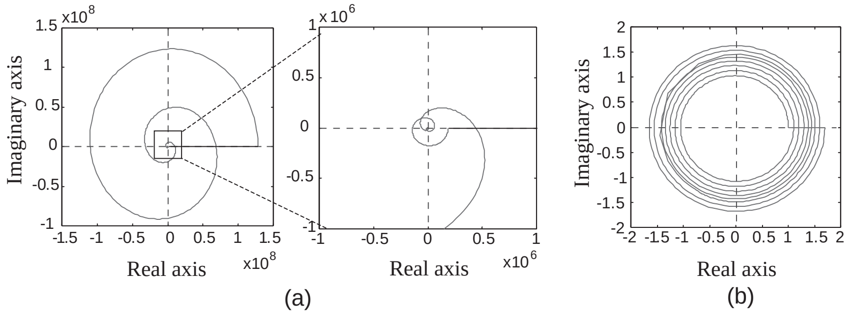

Fig. 4. Contour of $p(S)$ with 1000 checking points: (a) not scaled; (b) scaled.

Table 4
Coefficients $\bar{a}_i$, diagonal elements d_{ii} of **D**, and signs of d_i

i	$\rho = 1.005\lvert\lambda_{16}\rvert = 1.8109$			
	$\bar{a}_i$	d_{ii}	Sign of d_i	V
0	1.0000e−002	–	+	–
1	1.4034e−001	2.0691e+006	+	–
2	2.6190e+000	2.0691e+006	+	–
3	2.0305e+001	2.0691e+006	+	–
4	1.3467e+002	2.0690e+006	+	–
5	6.3241e+002	2.0663e+006	+	–
6	2.4684e+003	2.0543e+006	+	–
7	7.7049e+003	1.9066e+006	+	–
8	2.0299e+004	1.7472e+006	+	–
9	4.4632e+004	9.1824e+005	+	–
10	8.3822e+004	7.8539e+005	+	–
11	1.3349e+005	7.7102e+004	+	–
12	1.8217e+005	7.1261e+004	+	–
13	2.1093e+005	−1.1134e+004	−	1
14	2.0769e+005	−6.6954e+003	+	2
15	1.7086e+005	2.1307e+002	+	–
16	1.1666e+005	1.7810e+002	+	–
17	6.3635e+004	−3.4793e+000	−	3
18	2.7036e+004	−2.6561e+000	+	4
19	7.9432e+003	4.5488e−003	+	–
20	1.4384e+003	3.6052e−003	+	–

V represents the number of sign changes of d_i, '+' means positive value and '–' means negative value. The d_0 is defined as 1.

damping given by $\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ and concentrated dampers resulting in non-proportional damping matrix. All the eigenvalues are calculated by the Lanczos method developed by Kim and Lee (1999) and their radii from the origin in the complex plane are calculated by $\rho_i = |\lambda_i|$ as in Table 5.

The number of considered eigenvalue is 8 and the radius of the contour S and the open circle is calculated by the 1.005 times the magnitude of the eighth eigenvalue ($\rho = 1.005|\lambda_8| = 51.4075$). The half-circle of the contour S is initially divided into 24 equal points. And since the argument of the largest eigenvalue is 91.54°, the part of the contour between 86.09° and 93.91° is subdivided into seven equal parts as shown in Fig. 6. The total variation of the argument is 1440°. And, the number of rotations is

Table 5
Calculated eigenvalues

Mode number	Eigenvalues (λ)		Radius ($\rho = \|\lambda\|$)
	Real	Imaginary	
1,2	−1.1369	±46.2187	46.2327
3,4	−1.1369	±46.2187	46.2327
5,6	−1.3731	±51.1333	51.1517
7,8	−1.3731	±51.1333	51.1517
9,10	−3.3902	±81.0872	81.1490
11,12	−3.3902	±81.0872	81.1490
13,14	−3.9407	±87.4771	87.5659
15,16	−3.9407	±87.4771	87.5659
17,18	−8.1642	±127.4394	127.7006
19,20	−8.1642	±127.4394	127.7006
21.22	−10.2629	±142.8367	143.2049
23,24	−10.2629	±142.8367	143.2049
25,26	−14.8662	±171.7301	172.3720
27,28	−14.8662	±171.7301	172.3720
29,30	−20.5387	±201.6249	202.6683
31,32	−20.5387	±201.6249	202.6683
33,34	−23.7699	±216.7332	218.0328
35,36	−23.7699	±216.7332	218.0328

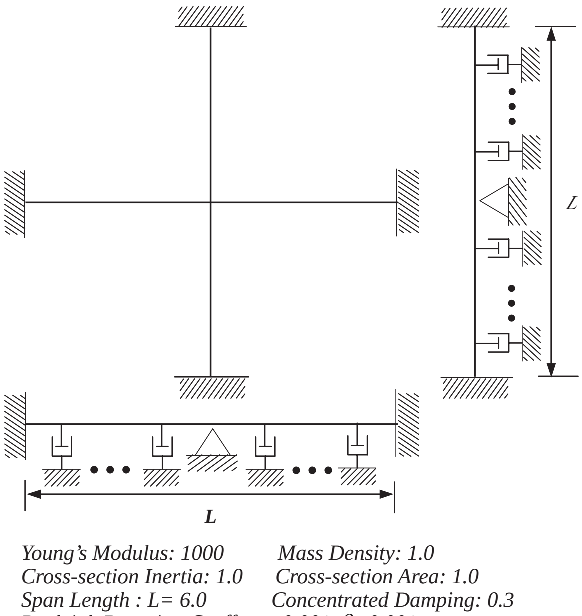

Fig. 5. Plane frame structure with lumped dampers.

$$N = \frac{\sum \Delta\theta_j}{2\pi} = \frac{1440^\circ}{360^\circ} = 4.$$

So, the number of eigenvalues in a circle of radius $\rho(= 1.005|\lambda_8|)$ is $8(= 4 \times 2)$, which exactly agree with the calculated values. If we assume that the largest eigenvalue is unknown and use the same radius for

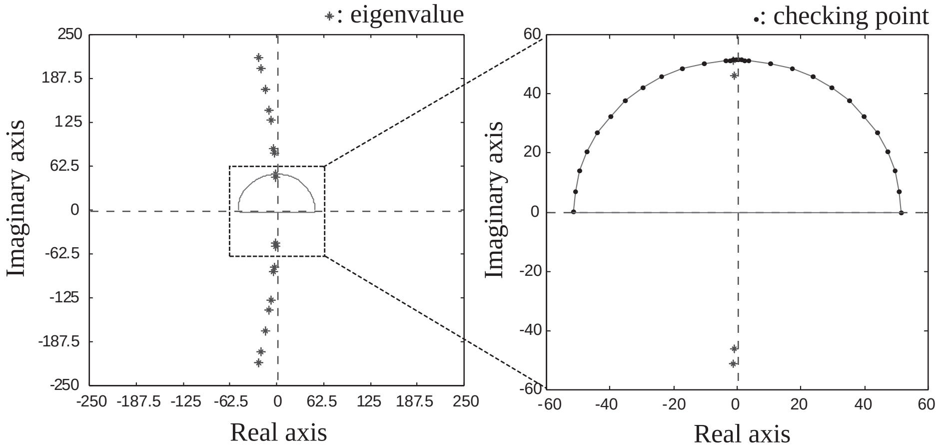

Fig. 6. Contour of S.

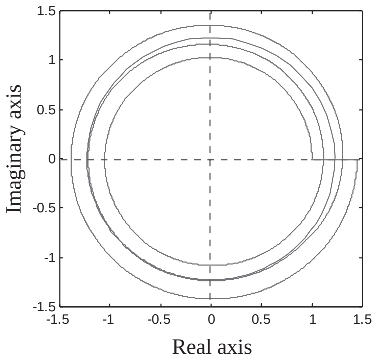

Fig. 7. Scaled contour of $p(S)$ with 1000 checking points.

the contour S, it is very difficult to detect the checking point where the drastic change of the variation of the argument occurs in this case. The change of the variation of the argument is 85.44° from the checking point at $\rho\angle 86.09°$ to the checking point at $\rho\angle 93.91°$. Because this value is smaller than 180°, we cannot conclude whether new checking points are needed or not. The scaled contour of $p(S)$ with 1000 checking points is shown in Fig. 7. The number of rotations for the contour $p(S)$ in the figure is 4.

The results for the modified Sturm sequence property are shown in Table 6. As shown at the last column in Table 6, the number of sign changes is 28. So if we use Eq. (21), the number of eigenvalues in the circle is $36 - 28 = 8$. Using Eq. (28), the number of positive elements in the matrix $\mathbf{D}$ is 8 as shown at the third column in Table 6. Therefore, we verify that the proposed method can exactly check the number of eigenvalues in some open disk of arbitrary radius for the system with multiple eigenvalues.

6. Conclusions

Methods of counting the number of eigenvalues for non-proportionally damped systems have been presented. The method based on argument principle requires many factorization processes at many

Table 6
Coefficients $\bar{a}_i$, diagonal elements d_{ii} of **D**, and signs of d_i

i	$\rho = 1.005\lvert\lambda_8\rvert = 51.4075$			
	$\bar{a}_i$	d_{ii}	Sign of d_i	V
0	2.5017e+006	–	+	0
1	2.3595e+006	−6.2584e+006	−	1
2	1.8523e+007	−6.2584e+006	+	2
3	1.5836e+007	−6.2584e+006	−	3
4	6.0360e+007	−6.2584e+006	+	4
5	4.6688e+007	−6.2584e+006	−	5
6	1.1462e+008	−6.2584e+006	+	6
7	8.0016e+007	−6.2584e+006	−	7
8	1.4176e+008	−6.2584e+006	+	8
9	8.9075e+007	−6.2584e+006	−	9
10	1.2111e+008	−6.2584e+006	+	10
11	6.8279e+007	−6.2584e+006	−	11
12	7.4043e+007	−6.2584e+006	+	12
13	3.7327e+007	−6.2584e+006	−	13
14	3.3137e+007	−6.2583e+006	+	14
15	1.4882e+007	−6.2576e+006	−	15
16	1.1017e+007	−6.2563e+006	+	16
17	4.3882e+006	−6.2388e+006	−	17
18	2.7450e+006	−6.2202e+006	+	18
19	9.6439e+005	−6.0086e+006	−	19
20	5.1461e+005	−5.9203e+006	+	20
21	1.5833e+005	−4.8338e+006	−	21
22	7.2533e+004	−4.7180e+006	+	22
23	1.9357e+004	−2.4454e+006	−	23
24	7.6408e+003	−2.4029e+006	+	24
25	1.7448e+003	−5.4284e+005	−	25
26	5.9414e+002	−5.3645e+005	+	26
27	1.1381e+002	−6.1754e+003	−	27
28	3.3379e+001	−4.1125e+003	+	28
29	5.1996e+000	2.4923e+004	+	–
30	1.3074e+000	1.3928e+004	+	–
31	1.5712e−001	3.0169e+003	+	–
32	3.3582e−002	2.8778e+003	+	–
33	2.8092e−003	2.5915e+000	+	–
34	5.0338e−004	2.5924e+000	+	–
35	2.2413e−005	2.8473e−004	+	–
36	3.2943e−006	1.6343e−004	+	–

V represents the number of sign changes of d_i, '+' means positive value and '−' means negative value. The d_0 is defined as 1.

checking points and sometimes it is difficult to detect drastic change of the variation of the argument. On the other hand, the proposed method needs only one factorization of the Schur–Cohn matrix. The final checking of the method is done by counting positive diagonal elements of the factorized Schur–Cohn matrix, which is very similar to the well known Sturm sequence property. By analyzing two numerical examples, it is verified that the proposed method can exactly calculate the number of eigenvalues in an open circle of given radius.

The proposed method is based on well-proven algorithms and theorems, however, during calculation of the coefficients of the characteristic polynomial, some small numerical errors may be accumulated mainly due to memory limitation. To apply the proposed method to large structures, therefore, further research to reduce the effects of numerical errors should be performed.

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